

TAUT SUBMANIFOLDS

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ABSTRACT. This is a short, elementary survey article about taut submanifolds. In order to simplify the exposition, we restrict to the case of compact smooth submanifolds of Euclidean or spherical spaces. Some new, partial results concerning taut 4-manifolds are discussed at the end of the text.

1. INTRODUCTION

This is a short, elementary survey article about taut submanifolds. In order to simplify the exposition, we restrict to the case of compact smooth submanifolds of Euclidean or spherical spaces. Sections 2 through 4 collect basic definitions and general results. In Section 5, we explain in greater detail some useful techniques that we use in Section 6 to prove some new, partial classification results about taut 4-manifolds.

2. THE SPHERICAL TWO-PIECE PROPERTY

Before introducing the concept of tautness, it is instructive to discuss the STPP. Let M be a compact surface embedded in an Euclidean sphere S^m . We say M has the *spherical two-piece property*, or STPP for short, if $M \cap B$ is connected whenever B is a closed ball in S^m [Ban70]. The STPP is equivalent to requiring that every Morse distance function of the form

$$L_q : M \rightarrow \mathbf{R}, \quad L_q(x) = d(x, q)^2$$

has exactly one local minimum. Intuitively, this can be seen by remarking that the sublevel sets of the L_q are exactly the closed balls B . Moreover, by replacing q by $-q$, one sees that these conditions imply that every Morse distance function L_q also has exactly one local maximum. Let b_i be the i th Betti number of M where, unless explicitly stated, we will always use \mathbf{Z}_2 coefficients to simplify the exposition, and let μ_i be the number of critical points of a Morse distance function L_q . From the Morse relations for the Euler characteristic,

$$\chi = b_0 - b_1 + b_2 = \mu_0 - \mu_1 + \mu_2,$$

one finally deduces that the STPP is equivalent to the fact that $\mu_i = b_i$ for all i and for all q such that L_q is a Morse function.

Since the STPP is defined in terms of intersections with closed balls, it is immediate that it is a conformally invariant property of compact surfaces in S^m . Of course, one can also consider the STPP for compact surfaces of Euclidean space. Since stereographic projection is a conformal transformation, the theories in S^m and \mathbf{R}^m are equivalent.

It follows from results of Kuiper and Banchoff (see [Ban70]) that a compact surface substantially embedded in S^m with the STPP is a round sphere or a cyclide of Dupin in S^3 , or the Veronese embedding of a projective plane in S^4 . Recall that an embedding into a sphere

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is called *substantial* if its image does not lie in a sphere of smaller dimension. Of course, it is enough for us to consider only substantial embeddings.

We finally note that the STPP condition is also equivalent to requiring that the induced homomorphism

$$H_0(M \cap B, \mathbf{Z}_2) \rightarrow H_0(M, \mathbf{Z}_2)$$

in Čech homology is injective for every closed ball B . The use of Čech homology allows one to use all closed balls rather than only those determined by level sets of distance functions that are Morse functions. If one prefers to use singular homology, then the equivalent condition is reformulated in terms that the above homomorphism in singular homology be injective for *almost* every closed ball.

3. TAUT SUBMANIFOLDS

Now we come to the main concept in this text. Let M be a compact submanifold embedded in S^m . We say M is *taut* if the induced homomorphism

$$(1) \quad H_i(M \cap B, \mathbf{Z}_2) \rightarrow H_i(M, \mathbf{Z}_2)$$

in Čech homology is injective for every closed ball B and for all i . This is equivalent to requiring that every Morse distance function L_q satisfies $\mu_i = b_i$ for all i , i.e. L_q is \mathbf{Z}_2 -perfect. Taut submanifolds were first considered by Carter and West [CW72]. It is clear that tautness is equivalent to the STPP in the case of surfaces. In the case of 3-manifolds, there is only a classification of the diffeomorphism types of manifolds admitting taut embeddings, which was given by Pinkall and Thorbergsson [PT89]. They remark that a geometrical classification, if possible, is expected to be a complicated problem since most of the examples already known depend on many parameters. Their result is that there are seven types of such 3-manifolds, namely S^3 , $\mathbf{R}P^3$, S^3/Q , $S^1 \times S^2$, $S^1 \times \mathbf{R}P^2$, $S^1 \times_h S^2$ and T^3 . Here $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group, and h is a diffeomorphism of $S^1 \times S^2$ that acts on each factor as the antipodal map. A classification of 4-manifolds admitting taut embeddings along similar lines seems feasible, and we will come back to this point in the last section.

Next we present other classes of examples of taut submanifolds. Clifford tori and the standard embeddings of projective spaces are important examples. We will not justify this assertion now, since these examples will be generalized below. It follows from the Chern-Lashof theorem [CL57] that the sphere S^n can only be tautly embedded in Euclidean space as a round hypersphere in an affine subspace. This result also admits an alternative proof for $n \geq 2$ by noting that the homology of S^n is trivial except in dimensions 0 and n , which implies that a Morse distance function to a taut embedding of S^n can have only critical points of index 0 and n , and so a focal point is necessarily of multiplicity n , which yields that S^n is umbilic according to an argument sketched below in Section 5 (compare [NR72]). Moving towards manifolds with more complicated topology, Cecil and Ryan proved that a taut n -dimensional compact hypersurface of S^{n+1} with the same homology as $S^k \times S^{n-k}$ has precisely two principal curvatures at each point, and the principal curvatures are constant along the corresponding curvature distributions [CR78]. They called such a hypersurface a (high-dimensional) *cyclide of Dupin*. Thorbergsson considered the case of a taut compact submanifold of dimension $2k$ that is $(k-1)$ -connected but not k -connected, and showed that it must be either a cyclide of Dupin diffeomorphic to $S^k \times S^k$ or the standard embedding of a projective plane over one of the four normed division algebras [Tho83b].

In another development, Bott and Samelson proved that the orbits of the isotropy representations of the symmetric spaces, sometimes called *s-representations*, are tautly embedded by explicitly constructing cycles forming a basis in \mathbf{Z}_2 -homology for these orbits [BS58]. These orbits are known as the *generalized flag manifolds*. The generalized flag manifolds are homogeneous examples of another very important, more general class of submanifolds, called *isoparametric submanifolds*. Hsiang, Palais and Terng proved that isoparametric submanifolds and their focal submanifolds are taut [HPT88].

4. TAUT REPRESENTATIONS

Most of the known examples of taut embeddings are homogeneous spaces. On the other hand, Thorbergsson derived some necessary topological conditions for a homogeneous space to admit a taut embedding, which allowed him to exhibit some examples of homogeneous spaces which cannot be tautly embedded, e.g. the lens spaces distinct from the real projective space [Tho88].

Thorbergsson and I approached the problem of classifying taut compact submanifolds of Euclidean space which are extrinsically Riemannian homogeneous by first studying *taut representations*, namely those representations of compact Lie groups all of whose orbits are tautly embedded. The only examples known at that time were the *s-representations*, and many proofs had been given of the tautness of special cases of generalized flag manifolds where the arguments were easier. In [GT03] (see also [GT00]), we classified the taut irreducible representations of the compact Lie groups. It turns out that the classification includes three families of representations that are not *s-representations*, thereby supplying many new examples of tautly embedded homogeneous spaces. These families are given by the following table, where $n \geq 2$.

| | |
|--|---|
| $\mathbf{SO}(2) \times \mathbf{Spin}(9)$ | (standard) $\otimes_{\mathbf{R}}$ (spin) |
| $\mathbf{U}(2) \times \mathbf{Sp}(n)$ | (standard) $\otimes_{\mathbf{C}}$ (standard) |
| $\mathbf{SU}(2) \times \mathbf{Sp}(n)$ | (standard) ³ $\otimes_{\mathbf{H}}$ (standard) |

It is worth noting that these representations are exactly those representations of cohomogeneity three that are not *s-representations*. In [GT02], we showed that the orbits of these representations also admit cycles of Bott-Samelson type.

The proof of the classification theorem in [GT03] is long and intricate. It starts with a remark by Kuiper that implies that a taut irreducible representation has the property that the second osculating spaces of all of its nontrivial orbits coincide with the ambient space [Kui61]. We call representations with this property, irreducible or not, *of class \mathcal{O}^2* . The class \mathcal{O}^2 is much more easy to deal with since it involves an infinitesimal condition. Indeed, we establish necessary upper bounds for the Dadok invariant $k(\lambda)$ (which is an integral algebraic invariant of an irreducible representation, see [Dad85]) for irreducible representations of class \mathcal{O}^2 , which allows us to reduce a lot the size of that class in such a way that the remaining cases are treated with geometrical methods that we develop in the second part of the proof.

There are three main strategies involved in these methods. The first one often works a kind of induction. It states that any slice representation of a taut representations inherits the property of being taut. (Recall that the *slice representation* of a representation at a point p is the representation induced by the isotropy subgroup at p on the normal space to the orbit through p at p .) The second one might be called “the fundamental result about taut sums”. It relates the topology of the orbits of a taut reducible representation to the topology

of the orbits of its summands. Finally, the third strategy invokes a reduction principle in transformation groups which *grosso modo* reduces the task of deciding whether a given representation is taut or not to the study of the tautness of a much simpler representation with trivial principal isotropy subgroup.

The combination of the above three strategies is also effective in the classification of taut reducible representations of compact *simple* Lie groups which I completed in [Gor04]. In this case, since every orbit of a summand of a taut reducible representation is also an orbit of the sum, it follows from the classification in the irreducible case that every irreducible summand of a taut reducible representation is an *s*-representation, and we need only to decide which sums of *s*-representations remain taut. Each compact simple Lie group admits few *s*-representations, so that the analysis via the geometrical methods discussed in the previous paragraph can be effectively carried out and the final result is given in the following table.

| | | |
|--------------------------------------|--|--|
| $\mathbf{SU}(n), n \geq 3$ | $\mathbf{C}^n \oplus \cdots \oplus \mathbf{C}^n$ | k copies, where $1 < k < n$ |
| $\mathbf{SO}(n), n \geq 3, n \neq 4$ | $\mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n$ | k copies, where $1 < k$ |
| $\mathbf{Sp}(n), n \geq 1$ | $\mathbf{C}^{2n} \oplus \cdots \oplus \mathbf{C}^{2n}$ | k copies, where $1 < k$ |
| \mathbf{G}_2 | $\mathbf{R}^7 \oplus \mathbf{R}^7$ | — |
| $\mathbf{Spin}(6)$ | $\mathbf{R}^6 \oplus \mathbf{C}^4$ | $\mathbf{R}^6 = (\text{vector}), \mathbf{C}^4 = (\text{spin})$ |
| $\mathbf{Spin}(7)$ | $\mathbf{R}^7 \oplus \mathbf{R}^8$ $\mathbf{R}^8 \oplus \mathbf{R}^8$ $\mathbf{R}^8 \oplus \mathbf{R}^8 \oplus \mathbf{R}^8$ $\mathbf{R}^7 \oplus \mathbf{R}^7 \oplus \mathbf{R}^8$ | $\mathbf{R}^7 = (\text{vector}), \mathbf{R}^8 = (\text{spin})$ |
| $\mathbf{Spin}(8)$ | $\mathbf{R}_0^8 \oplus \mathbf{R}_+^8$ $\mathbf{R}_0^8 \oplus \mathbf{R}_0^8 \oplus \mathbf{R}_+^8$ $\mathbf{R}_0^8 \oplus \mathbf{R}_0^8 \oplus \mathbf{R}_0^8 \oplus \mathbf{R}_+^8$ | $\mathbf{R}_0^8 = (\text{vector}), \mathbf{R}_+^8 = (\text{halfspin})$ |
| $\mathbf{Spin}(9)$ | $\mathbf{R}^{16} \oplus \mathbf{R}^{16}$ | $\mathbf{R}^{16} = (\text{spin})$ |

We make two simple remarks regarding the representations appearing in this table. The first one is that in the case of the first representation in the table, any number of summands \mathbf{C}^n in the sum can be replaced by the dual representation $(\mathbf{C}^n)^*$, and the resulting representation remains taut. The second one is that the representations of $\mathbf{Spin}(8)$ are listed up to composition with an outer automorphism of the Lie group, so the pair $(\mathbf{R}_0^8, \mathbf{R}_+^8)$ appearing in the list can be replaced by any pair of inequivalent 8-dimensional representations of $\mathbf{Spin}(8)$, and the resulting representations for $\mathbf{Spin}(8)$ will still be taut.

Despite these results, the classification of taut extrinsically Riemannian homogeneous submanifolds of Euclidean space is still far from complete. For example, we still do not know what the taut reducible representations of the nonsimple groups are.

5. THE MORSE INDEX THEOREM, PROPER DUPIN SUBMANIFOLDS, AND OZAWA'S THEOREM

In this section, we review a collection of methods related to taut submanifolds that were used in [PT89] and will be useful here.

Let M be a compact smooth n -manifold substantially embedded in S^{n+k} . Later we will specify to the case in which M is tautly embedded. Denote by $N^1(M)$ the unit normal bundle of M in S^{n+k} . A focal point of M in S^{n+k} is a critical value of the restriction of the exponential map of S^{n+k} to $N^1(M)$. According to this definition, $-p$ is a focal point of M for every $p \in M$ if $k > 1$. This kind of focal point is uninteresting for us since it does not relate

to the geometry of M , but rather to the fact that any pair of antipodal points are conjugate in the sphere. Therefore in the following we will completely disconsider this kind of focal point. All the other focal points of M can be described as follows. For $\xi \in N^1(M)$, the set of focal points of M lying in the normal ray defined by ξ is discrete and in correspondence with the principal curvatures of M relative to the Weingarten operator A_ξ in such a way that for a focal distance $d \in (0, \pi)$ there corresponds an eigenvalue $\cot d$ of A_ξ .

Now it can be checked that a point $q \in S^{n+k}$ is a focal point of M if and only if the distance function L_q is not a Morse function. In this case, q is a focal point relative to some $\xi \in N_p^1(M)$, p is a degenerate critical point of L_q , and the multiplicity of q as a focal point relative to ξ equals the nullity of the Hessian of L_q at p , which is also the multiplicity of the corresponding principal curvature of A_ξ . For a nonfocal point q , L_q is a Morse function and the Morse index theorem asserts that the index of a critical point p of L_q equals the sum of the multiplicities of the focal points of M lying in the normal geodesic segment \overline{pq} .

It follows that the sum of the multiplicities of the focal points lying in any open half great circle joining p and $-p$ is n . Since $A_{-\xi} = -A_\xi$ and $-\cot d = \cot(\pi - d)$, we also have that for each focal distance $d \in (0, \pi)$ in the direction of ξ , there is a corresponding focal distance $\pi - d$ in the direction of $-\xi$ with the same multiplicity. Let $n(\xi)$ denote the number of distinct focal points lying in the open half great circle specified by ξ . We have that $n(\xi)$, as a function on $N^1(M)$, is lower semicontinuous and there is an open and dense subset Ω of $N^1(M)$ where $n(\xi)$ is locally constant and maximal. A normal vector $\xi \in N^1(M)$ is called *regular* if it belongs to Ω , and *singular* otherwise.

We suppose henceforth that M is tautly embedded in S^{n+k} .

Now the condition that every unit normal vector is regular is equivalent to the condition that M be a *proper Dupin* submanifold of S^{n+k} [Pin86, Tho83a]. The latter means that the principal curvatures are constant along the corresponding curvature surfaces and the number of distinct principal curvatures is constant. In this case, there is a well defined ordered sequence (m_1, \dots, m_g) of multiplicities of focal points of M along the normal ray defined by an arbitrary ξ . Here g is the (constant) number of distinct principal curvatures of M .

If M is taut and there exists a unit normal vector $\xi \in N^1(M)$ such that $n(\xi) = 1$, then M is umbilic, and hence, a great hypersphere S^n in S^{n+1} . This can be seen as follows. Let f be the first focal point of M in the direction of ξ . The multiplicity of f is n by hypothesis. Consider the geodesic segment \overline{pf} , the open half great circle γ containing it, and a point q belonging to γ . The Morse index theorem implies that the index of p as a critical point of L_q is 0 or n , according to whether q occurs between p and f or past f . It follows that p is a local minimum or a local maximum of L_q , respectively. By tautness of M , there can be only one local minimum (resp. local maximum), so this is also a global minimum (resp. global maximum). Letting q tend to f in both cases shows that M is contained in the hypersphere of center p and radius $d(p, f)$, and hence M coincides with that hypersphere.

As was said above, if M is taut and q is a focal point, then L_q is not a Morse function, but there is a very useful theorem by Ozawa [Oza86] that asserts that L_q is a \mathbf{Z}_2 -perfect Morse-Bott function and the connected components of the critical set of L_q are compact smooth taut submanifolds of S^{n+k} . Recall that a function F is called a *Morse-Bott* function if the connected components of its critical set are smooth submanifolds along each of which the nullity of the Hessian of F is constant and equal to the dimension of the component [Bot54]. In this case, the index of F along a critical manifold is by definition the index of the Hessian

of F restricted to the normal bundle of that critical manifold. A Morse-Bott function F on M is *perfect* if it satisfies the Morse-Bott equalities, namely the Poincaré polynomials of M and those of the critical manifolds C_i of F are related by the formula

$$\mathcal{P}_t(M) = \sum_i \mathcal{P}_t(C_i) t^{\text{ind} C_i}.$$

If M is taut, it follows from Ozawa's theorem and the Morse-Bott equalities that the \mathbf{Z}_2 -homology of the critical manifold corresponding to the set of minima (resp. maxima) of a distance function L_q injects into the \mathbf{Z}_2 -homology of M . Such critical sets are called *top sets* or *top cycles*, or yet *top circles*, *top tori*, etc., when their topology is specified, following terminology introduced by Kuiper and expanded by Pinkall and Thorbergsson. For each $\xi \in N^1(M)$, we denote by $T(\xi)$ the top cycle which is the set of minima of L_f , where f is the first focal point in the normal ray defined by ξ .

Another interesting consequence of Ozawa's theorem is that the definition of a taut submanifold can be restated so as to require that the homomorphism (1) in *singular* homology be injective for *all* closed balls B .

6. TAUT 4-MANIFOLDS

In this section we present some new, partial results about the classification of compact smooth manifolds of dimension four that admit taut embeddings. We start with the two following results, the first one of which has already been noticed by Thorbergsson.

Theorem 1. *A compact four-dimensional smooth taut submanifold M with vanishing first Betti number is diffeomorphic to S^4 , $S^2 \times S^2$ or \mathbf{CP}^2 .*

Proof. If $b_2 = 0$, then M is a \mathbf{Z}_2 -homology sphere, and we have already seen that it must be a hypersphere S^4 in S^5 .

Next, we remark that a taut submanifold with $b_1 = 0$ must be simply connected. This is because it admits a \mathbf{Z}_2 -perfect Morse function, and such a function has no critical points of index one (compare Lemma 4.11(1) in [CE75]).

If $b_2 \neq 0$, it follows from the preceding remark that M is a taut 4-manifold that is 1-connected but not 2-connected. Therefore Theorem A in [Tho83b] says that either M is a cyclide of Dupin of type $S^2 \times S^2$ in S^5 , or M is diffeomorphic to \mathbf{CP}^2 and sits inside S^7 . Recall that the standard embedding of \mathbf{CP}^2 in S^7 is taut. \square

Theorem 2. *A compact four-dimensional smooth taut submanifold M with vanishing second Betti number is diffeomorphic to S^4 or $S^1 \times S^3$.*

Proof. By Theorem 1, we may assume that $b_1 \neq 0$. Then, by hypothesis and \mathbf{Z}_2 -Poincaré duality, exactly four \mathbf{Z}_2 -homology groups of M are nonzero. It follows from Theorem 3.2 in [Oza86] that $b_1 = 1$ (this also follows from Theorem 11 in [Heb88]).

We will use some basic facts about the topology of 4-manifolds [GS99]. Since there is a Morse function on M with one critical point of each index 0, 1, 3, 4, there is a handle decomposition $M = h_0 \sqcup h_1 \sqcup h_3 \sqcup h_4$, where each h_i an i -handle attached via an attaching map.

If M is orientable, then h_1 is an orientable handle, $h_0 \sqcup h_1$ and $h_3 \sqcup h_4$ are both diffeomorphic to $S^1 \times D^3$, and M is obtained by glueing two copies of $S^1 \times D^3$ along their boundaries by a diffeomorphism f of $S^1 \times S^2$. The diffeomorphism class of M only depends on the diffeotopy class of f . It has been shown that the diffeotopy group of $S^1 \times S^2$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$,

where each generator extends to a diffeomorphism of $S^1 \times D^3$ [Glu62]. It follows easily that there is at most one manifold in this case, which plainly is $S^1 \times S^3$.

If M is nonorientable, then h_1 is a nonorientable handle, $h_0 \sqcup h_1$ and $h_3 \sqcup h_4$ are both diffeomorphic to $S^1 \tilde{\times} D^3$, the twisted 3-disk bundle over S^1 , and M is obtained by glueing two copies of $S^1 \tilde{\times} D^3$ along their boundaries via a diffeomorphism of $S^1 \tilde{\times} S^2$. Every diffeomorphism of $S^1 \tilde{\times} S^2$ is diffeotopic to one that extends to $S^1 \tilde{\times} D^3$ [KR90]. It follows as above that M must be diffeomorphic to $S^1 \tilde{\times} S^3$, but the following lemma shows that this case cannot occur. This finishes the proof. \square

Lemma 1. *We have that $S^1 \tilde{\times} S^3$ does not admit a taut embedding.*

Proof. Suppose, on the contrary, that $M = S^1 \tilde{\times} S^3$ admits a taut and substantial embedding into a sphere S^{4+k} . Since $b_2 = 0$, any top set must have vanishing second Betti number. It follows that M cannot admit a 2-dimensional top set, and any 3-dimensional top set has the \mathbf{Z}_2 -homology of S^3 . Therefore, any top set of M , being taut, is diffeomorphic to S^1 or S^3 . This implies that there is no $\xi \in N^1(M)$ with $n(\xi) = 4$. For otherwise, if q is a point lying in between the second and third focal points in the normal ray defined by ξ , and p is the foot point of ξ , then p is a critical point of index 2 of L_q , but $b_2 = 0$.

Now, for every ξ , $n(\xi) = 2$ or 3. We claim that M is proper Dupin. If not, there exist a singular $\bar{\xi}$ with $n(\bar{\xi}) = 2$ and a sequence ξ_i converging to $\bar{\xi}$ with $n(\xi_i) = 3$. By replacing $\bar{\xi}$ and the ξ_i by their opposites if necessary, we may assume that the top set $T(\bar{\xi})$ is diffeomorphic to S^3 . The $T(\xi_i)$ are round circles representing nontrivial one-dimensional homology classes of M . A subsequence of the $T(\xi_i)$ converges to a circle in $T(\bar{\xi})$ that is homologically nontrivial, and this is a contradiction (compare the proof of Theorem 2.2 in [Oza86]).

Now M is proper Dupin. Suppose first that $n(\xi) = 2$ for all ξ . We cannot have $k = 1$, since M is not a cyclide of Dupin, so $k > 1$. Note that the multiplicities of the principal curvatures are 1 and 3. Let T be a small tube around M . Then T is a proper Dupin hypersurface with three distinct principal curvatures of multiplicities 1, 3 and $k - 1$. It is known that a proper Dupin hypersurface with 3 distinct principal curvatures has all multiplicities equal [Miy84], so this case cannot occur.

Suppose now that $n(\xi) = 3$ for all ξ . A small tube T around M is a proper Dupin hypersurface with 4 distinct principal curvatures of multiplicities $(1, 2, 1, k - 1)$. It is known that in the case of 4 distinct principal curvatures there are at most two different multiplicities [GH91]. It follows that $k = 3$. Let B_+ and B_- be the two components of $S^7 \setminus T$ and assume that $M \subset B_+$. It follows from the Mayer-Vietoris sequence that

$$H_3(T; \mathbf{Z}) = H_3(B_+; \mathbf{Z}) \oplus H_3(B_-; \mathbf{Z}).$$

Since B_+ is homotopy equivalent to M , we see that $H_3(T; \mathbf{Z}) = H_3(M; \mathbf{Z}) \oplus H_3(B_-; \mathbf{Z}) = \mathbf{Z}_2 \oplus H_3(B_-; \mathbf{Z})$ (see [Hat02], p. 238). On the other hand, by [GH91], $H_3(T; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. This is a contradiction. Thus there can be no taut embedding. \square

Remarks 1. (i) As a consequence of Theorem 2, a compact smooth 4-manifold with the same homology as $S^1 \times S^3$ which is tautly embedded in arbitrary codimension is diffeomorphic to $S^1 \times S^3$ (compare Theorem 2 in [CR78] and the next item (ii)).

(ii) According to Theorem 3.1 in [Oza86], the codimension of a taut and substantial embedding of $S^1 \times S^3$ into a sphere is either 1 or 3. Both codimensions can be realized. In fact, $S^1 \times S^3$ can be realized as a cyclide of Dupin in codimension one, and it is shown in [CR85] that any taut embedding of $S^1 \times S^3$ with codimension one is such a cyclide, and

they are all Möbius equivalent to a tube around a circle in S^5 . Moreover, $S^1 \times S^3$ can also be tautly embedded in S^7 as a singular orbit of the isotropy representation of the Grassmann manifold $\mathbf{SO}(6)/(\mathbf{SO}(2) \times \mathbf{SO}(4))$.

(iii) We have that $S^1 \times_h S^3$, where h acts by the antipodal map on each factor, is diffeomorphic to $S^1 \times S^3$. In fact, $S^1 \times_h S^3 \approx \mathbf{U}(1) \times_{\mathbf{Z}_2} \mathbf{SU}(2) = \mathbf{U}(2)$ and $\det : \mathbf{U}(2) \rightarrow \mathbf{U}(1)$ is a principal $\mathbf{SU}(2)$ -bundle with a global section $\sigma : \mathbf{U}(1) \rightarrow \mathbf{U}(2)$ given by

$$\sigma(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) According to Theorem 8 in [Heb88], $S^2 \tilde{\times} S^2 \approx \mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ cannot be tautly embedded into a sphere.

Regarding the following conjecture, note that the principal orbits of the isotropy representation of the Grassmann manifold $\mathbf{SO}(5)/(\mathbf{SO}(2) \times \mathbf{SO}(3))$ are proper Dupin hypersurfaces in S^5 diffeomorphic to $S^1 \times \mathbf{RP}^3$.

Conjecture 1. *A compact embedded proper Dupin hypersurface in S^5 with four distinct principal curvatures is diffeomorphic to $S^1 \times \mathbf{RP}^3$.*

We next present a proof of this conjecture that uses that the Poincaré conjecture in dimension 3 is true (there has been recent progress by Perelman on this conjecture, see [Mor05]). Let M be a compact embedded proper Dupin hypersurface in S^5 with four distinct principal curvatures. Then the multiplicities of focal points along a normal ray are $(1, 1, 1, 1)$. For a fixed unit normal vector field, let F_+ (resp. F_-) denote the set of focal points corresponding to the largest (resp. smallest) principal curvature, and let $f_{\pm} : M \rightarrow F_{\pm}$ be the focal maps. Then these are circle bundles whose fibers consist of the circles of curvature. It follows from the proof of Proposition 3.5 in [GH91] that $\pi_1(F_+) = \mathbf{Z}_2$ and $\pi_1(F_-) = \mathbf{Z}$ or $\pi_1(F_+) = \mathbf{Z}$ and $\pi_1(F_-) = \mathbf{Z}_2$, and we may assume we are in the first case just by replacing the normal field by its opposite if necessary. Now the universal covering manifold of F_+ is a simply connected compact 3-manifold, hence diffeomorphic to S^3 if the Poincaré conjecture is true. The quotient of S^3 by a fixed point free involution is diffeomorphic to \mathbf{RP}^3 according to [Liv60], so F_+ is diffeomorphic to \mathbf{RP}^3 . So far we know that M is a circle bundle over \mathbf{RP}^3 . Besides the trivial one, there are exactly two other circle bundles over \mathbf{RP}^3 , one oriented, one nonoriented (in fact, the associated 2-plane bundles are $2L$ and $L \oplus T$, where T is the trivial line bundle and L is the tautological line bundle [Lev63]). Let E_1, E_2 respectively denote the total spaces of these bundles. Using the Gysin sequence in the oriented case, one easily computes that $H^2(E_1; \mathbf{Z}) = 0$, and hence, $H_2(E_1; \mathbf{Z}) = 0$. Using the Serre homology spectral sequence in the nonoriented case, it is also not difficult to see that $H_1(E_2; \mathbf{Z})$ is a \mathbf{Z}_2 -extension of \mathbf{Z}_2 . In any case, E_1 and E_2 do not have the integral homology of the trivial circle bundle $\mathbf{RP}^3 \times S^1$. But M must have the integral homology of $\mathbf{RP}^3 \times S^1$ according to [GH91]. It follows that M is diffeomorphic to $\mathbf{RP}^3 \times S^1$.

It is easy to construct other examples of taut embeddings of compact smooth 4-manifolds that are not covered by the previous results, so there is still a lot of work to be done in order to achieve a complete classification. In the case of taut hypersurfaces in S^5 with no more than three distinct principal curvatures at each point, we have the following theorem [Nie91].

Theorem 3 (Niebergall). *Let M be a compact smooth taut hypersurface in S^5 . If M has at most three distinct principal curvatures at each point, then M is diffeomorphic to one of the following 4-manifolds: S^4 , $S^3 \times S^1$, $S^2 \times S^2$, $T^2 \times S^2$, or the nontrivial S^2 -bundle over \mathbf{RP}^2 .*

Niebergall proves this theorem by first remarking that there is an open and dense subset M_0 of M where the multiplicities of the principal curvatures are locally constant. Then each connected component of M_0 is a piece of a proper Dupin hypersurface with three distinct principal curvatures in S^5 . By the main result of [Nie91], each connected component of M_0 is then a *reducible* proper Dupin hypersurface [Pin85], and hence can be reconstructed from lower dimensional proper Dupin hypersurfaces, where the possibilities for these are already known. The final step of the argument uses an unpublished result of Pinkall stating that a taut hypersurface is analytic.

We finish this text with the following lemma which refers to the case of a taut hypersurface in S^5 with generically four distinct principal curvatures.

Lemma 2. *A compact taut hypersurface M in S^5 with four distinct principal curvatures at some point has $b_i \geq 2$ for $i = 1, 2, 3$.*

Proof. Let $\xi \in N_p^1(M)$ with $n(\xi) = 4$. Then the multiplicities of the focal points in the direction of ξ are $(1, 1, 1, 1)$. Let q be a point in the normal ray issuing from p in the direction of ξ that comes after the second focal point and before the third one. Then L_q has a critical point of index 2 at p , so $b_2 \geq 1$. Upon a choice of unit normal vector field on M , there is an identification of $N(M)$ with $M \times \mathbf{R}$. Let $F : M \times (-\pi, \pi) \subset N(M) \rightarrow S^5$ be the restriction of the endpoint map. The focal set of M consists of the critical values of F and consists of focal varieties of codimension at least 2, hence its complement W is connected. Let $V = F^{-1}(W)$. Then $F|_V : V \rightarrow W$ is a local diffeomorphism. For $(x, t) \in V$, let $j(x, t)$ be the index of x as a critical point of L_q for $q = F(x, t)$. Then j is locally constant. Decompose V into a disjoint union of nonempty open subsets

$$V = V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4,$$

where $V_\iota = \{(x, t) \in V : j(x, t) = \iota\}$. If $b_2 = 1$, then $F : V_2 \rightarrow W$ is a diffeomorphism. This implies that V_2 is connected, but it is clear that V_2 must contain points on both sides of $M \times \{0\} \subset V_0$, so this is a contradiction. Hence, $b_2 \geq 2$. Similarly, $b_1 \geq 2$. \square

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